Lecture 6: Isospin and SU(3)

- The Hadronic Spectrum
- Isospin and Scattering Relations
- Resonances: The Δ
- Strangeness
- Charge and the Gell-Mann Nishijima Eq
- Group Theory Interpretation
- SU(2) and SU(3)
- Quark Model Interpretation

The Hadronic Spectrum

- $\exists$ only 3 generations of leptons (e, μ, τ and their respective neutrinos), but hundreds of hadrons
- Physicists soon realized that it's not sensible to consider these hadrons fundamental
  - Look for basic patterns in masses, spins, charges
  - Look for rules to relate interaction rates and decay rates of different hadrons in terms of internal quantum numbes
- Today we know hadrons are composite particles made of quarks
  - Spectrum of observed particles analog of period table of elements
  - Because α_s large at low mom transfer, the theory is not perturbative
  - We cannot calculate the wave functions of the quark bound states (hadrons)
  - We'll see in a few weeks that bound states of heavy quarks can give us clues to the shape of the potential
- In 1960's no one knew whether quarks were real or just mathematical constructs
  - But we'll use our modern knowledge to inform our discussion and terminology

Classification of Hadrons

- Mesons (integer spin) vs Baryons (half integer spin)
  - Baryons must be pair produced, but mesons can be produced singly
- Baryons
  - Earliest examples: p and n
  - Fact that both appear to see same nuclear force and that the masses are so close together ($m_p = 938.20$ MeV, $m_n = 939.57$ MeV) make it natural to think of them as 2 states of same particle: the nucleon N
  - Define isospin (with same algebra as spin: SU(2)). Then N has $I = \frac{1}{2}$:

$$p = \left|\frac{1}{2}\frac{1}{2}\right\rangle \quad n = \left|\frac{1}{2} - \frac{1}{2}\right\rangle$$

- Mesons
  - Earliest example: The pions
  - Three charges π^+, π^0 ,π^- so $I = 1$:

$$\pi^+ = |11\rangle \quad \pi^0 = |10\rangle \quad \pi^- = |1-1\rangle$$

- Note: For N, $Q = I_z + \frac{1}{2}$ while for π $Q = I_z$

  These are special cases of a more general rule we'll get to soon

Example: πN scattering

- Can use isospin to relate different reaction rates
- Each value of isospin that is possible provides an independent matrix element
- For πN scattering $I = 1 \otimes I = \frac{1}{2} \Rightarrow I = \frac{3}{2}, \frac{1}{2}$ so $\exists$ 2 indep matrix elements

$$\mathcal{M}_{\frac{1}{2}} \equiv \left\langle \frac{1}{2} \middle| H \middle| \frac{1}{2} \right\rangle \quad \mathcal{M}_{\frac{3}{2}} \equiv \left\langle \frac{3}{2} \middle| H \middle| \frac{3}{2} \right\rangle$$

- Examples of decomposition

$$\begin{aligned} p\pi^+ &= \left| \frac{3}{2} \frac{3}{2} \right\rangle \\ p\pi^0 &= \sqrt{\frac{2}{3}} \left| \frac{3}{2} \frac{1}{2} \right\rangle - \sqrt{\frac{1}{3}} \left| \frac{1}{2} \frac{1}{2} \right\rangle \end{aligned}$$

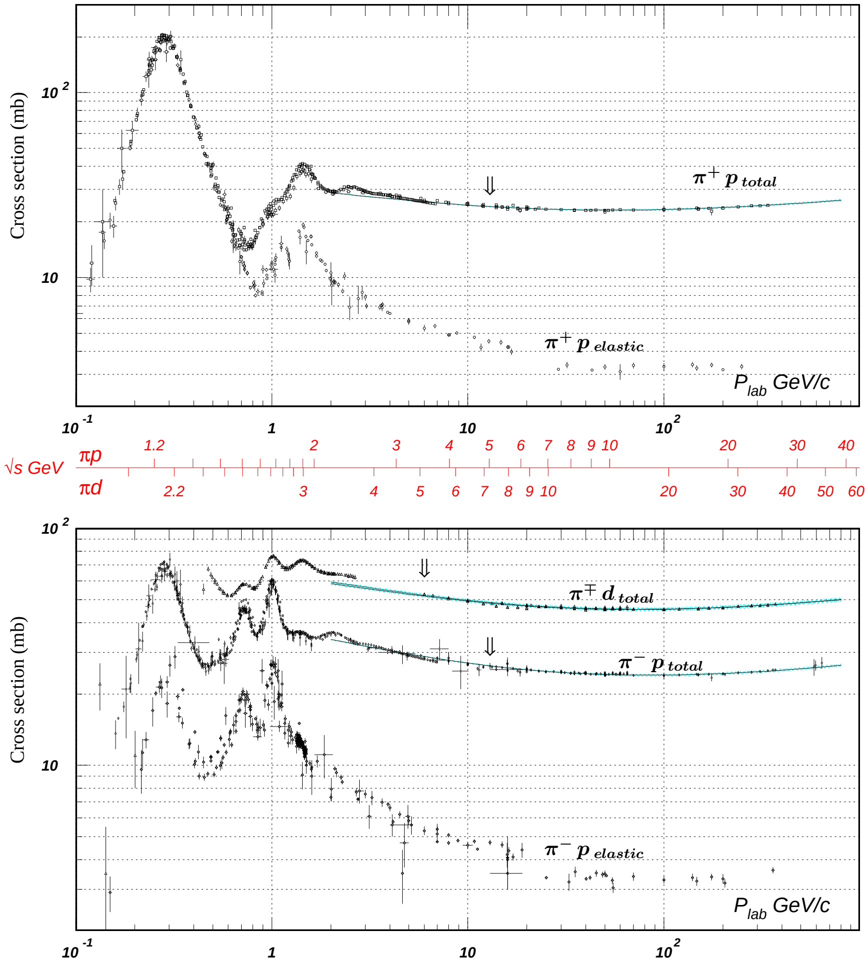
and so forth

- Thus

$$\begin{aligned} \sigma(\pi^+ p \to \pi^+ [) &\sim |\mathcal{M}_{\frac{3}{2}}|^2 \\ \sigma(\pi^+ n \to \pi^+ n) &\sim |\frac{1}{3}\mathcal{M}_{\frac{3}{2}} + \frac{2}{3}\mathcal{M}_{\frac{1}{2}}|^2 \\ \sigma(\pi^- p \to \pi^0 n) &\sim |\frac{\sqrt{2}}{3}\mathcal{M}_{\frac{3}{2}} - \frac{\sqrt{2}}{3}\mathcal{M}_{\frac{1}{2}}|^2 \end{aligned}$$

We can determine all the scattering rates in terms of these 2 amplitudes

More on pN scattering



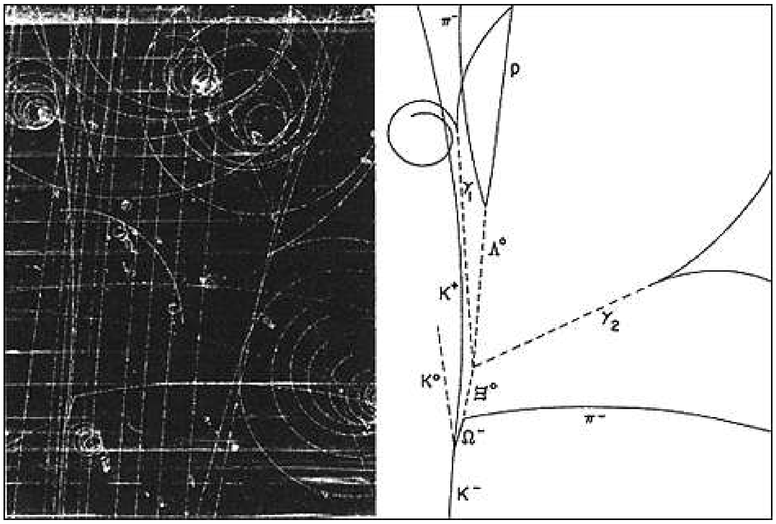
- Large bumps: “resonances”
- Eg: near 1236 MeV
  - Width $\sim$ 120 MeV $\Rightarrow$ short lifetime: $\Delta E \Delta t \sim \hbar$:

$$\Delta t \sim \frac{\hbar}{\Delta E} \sim \frac{6.58 \times 10^{-22} \text{ MeV sec}}{120 \text{ MeV}} \sim 5 \times 10^{-24} \text{ sec}$$

  This resonance is called the Δ

- Four states: $I = 3/2$: Δ^{++},Δ^{++},Δ^{+}, Δ^{0},Δ^{-}
- There is NO Δ^{--}

Strangeness



- In 1950's a new class of hadrons seen
  - Produced in πp interaction via Strong interactions
  - But travel measureable distance before decay, so decay is weak

Why should this happen? There must be conserved quantum number preventing the strong decay

- Example:

$$\pi^- p \longrightarrow \Lambda^0 K^0$$

  - $\Lambda^0 \rightarrow p\pi^-$ with lifetime $\tau = 2.6 \times 10^{-10}$ sec
  - $K^0 \rightarrow \pi^+\pi^-$ with lifetime $\tau = 0.8958 \times 10^{-10}$ sec

- Assign a new quantum number called strangeness to the Λ and K^0
- By convention Λ has $S = -1$ and K^0 has $S = 1$ (an unfortunate choice, but we are stuck with it)

Strangeness and I_Z

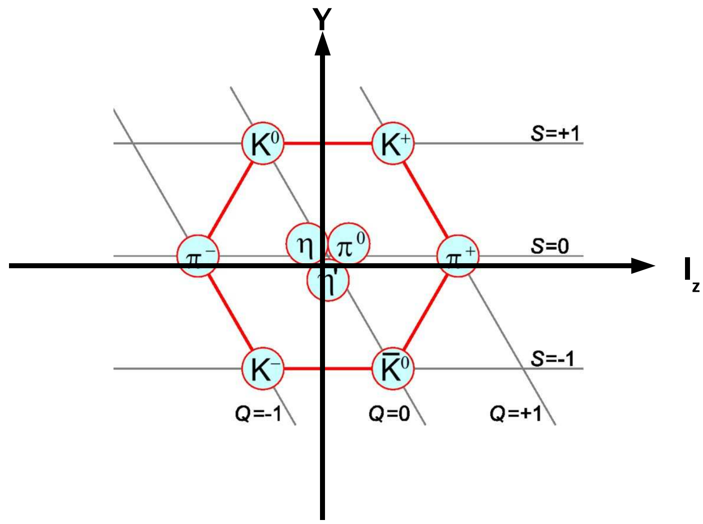
- We've already seen that within an isospin multiplet, different I_z have different charge
- Can generalize this observation for all light quark (u,d,s) multiplets:

$$Q = I_z + \frac{B+S}{2}$$

  Define hypercharge $Y \equiv B+S$

- This is called the Gell Mann-Nishijima Eq
- Note: Because Q is determined from I_3, EM interactions cannot conserve isospin, but do conserve I_3
  - This is analogous to the Zeeman effect in atomic physics where a B field in z direction destroys conservation of angular momentum, but leaves J_z as a good quantum number
- EM coupling $\sim 1\%$ so effects of isospin non-conservation are small and can be treated as perturbative correction to strong interaction

Group Theory Interpretation



- Describe particles with same spin, parity and charge congugation symmetry as members of a multiplet with different I_z and Y
- Will define (next 2 pages) raising and lowering operators to navigate around the multiplet
- Gell Man and Zweig suggested that patterns of multiplets could be explained if all hadrons were made of quarks
  - Mesons: $q\bar{q}$ $\quad 3 \otimes \bar{3} = 1 \oplus 8$
  - Baryons: qqq $\quad 3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$
- In those days, 3 flavors (extension to 6 discussed later)

Introduction to Group Theory (via SU(2))

- Let's start by reviewing SU(2) Isospin
- Fundamental representation: a doublet

$$\chi = \begin{pmatrix} u \\ d \end{pmatrix} \quad \text{so} \quad u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Define rotation in isospin space in terms of infinitesmal generators of the rotations

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The τ matrices satisfy commutation relations

$$[\frac{1}{2}\tau_i, \frac{1}{2}\tau_j] = i\,\varepsilon_{ijk}\tau_k$$

These commutation relations define the SU(2) algebra

- We can have higher representations of SU(2): $N \times N$ matrices with $N = 2I + 1$
- Also, there is an operator that commutes with all the τ's:

$$I^2 = (\frac{1}{2}\vec{\tau})^2 = \frac{1}{4}\Sigma_i\, \tau_i^2$$

and there are raising and lowering operators

$$\tau_\pm = \frac{1}{2}(\tau_1 \pm i\tau_2)$$

Extension to SU(3))

- SU(3): All unitary transformations on 3 component complex vectors without the overall phase rotation (U(1))

$$U^\dagger U = UU^\dagger = 1 \qquad \det U = 1$$

$$U = exp[i \sum_{a=1}^{8} \lambda_a \theta_a / 2]$$

- The fundamental representation of SU(3) are 3×3 matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- Commutation relations:

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2}$$

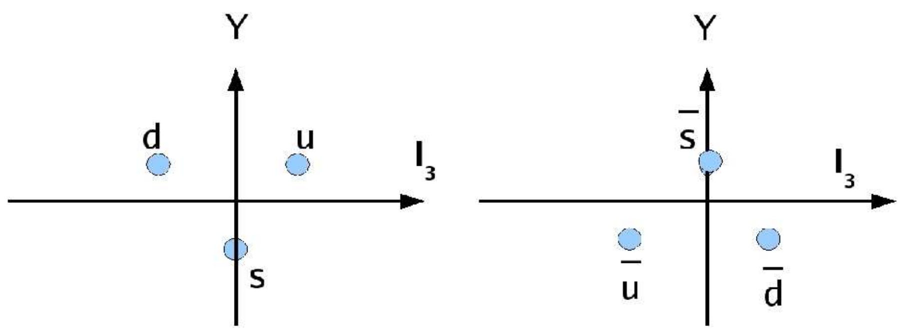
where $f_{123} = 1$, $f_{147} = f_{246} = f_{257} = f_{345} = \frac{1}{2}$, $f_{156} = f_{367} = -\frac{1}{2}$ and $f_{458} = f_{678} = \sqrt{3}/2$.

SU(3) Raising and Lowering Operators

- SU(3) contains 3 SU(2) subgroups embedded in it

$$\begin{array}{llll} \text{Isospin :} & F_1 & F_2 & F_3 \\ \text{U}-\text{spin :} & F_6 & F_7 & \sqrt{3}F_8 - F_3 \\ \text{V}-\text{spin :} & F_4 & F_5 & \sqrt{3}F_8 + F_3 \end{array}$$

- For each SU(2) subgroup we can form the usual raising and lowering operators
- Any two of the three subgroups are enough to navigate through all the members of the multiplet
- Fundamental representation: A triplet



- Define group structure of the state by starting at one corner and using raising and lowering operators

$$\begin{aligned} (V_-)^{p+1}\phi_{max} &= 0 \\ (I_-)^{1+1}\phi_{max} &= 0 \\ \text{structure :} & \ (p,q) \end{aligned}$$

- So quarks (u,d,s) have $p=1,\ q=0$ while antiquarks $(\overline{u},\overline{d},\overline{s})$ have $p=0,\ q=1$

Combining SU(3) states (2 quarks)

- Combining two SU(3) objects gives $3 \times 3 = 9$ possible states

$$
\begin{array}{ccc}
uu & & \\
\frac{1}{\sqrt{2}}(ud+du) & & \frac{1}{\sqrt{2}}(ud-du) \\
dd & & \\
\frac{1}{\sqrt{2}}(us+su) & & \frac{1}{\sqrt{2}}(us-su) \\
ss & & \\
\frac{1}{\sqrt{2}}(ds+sd) & & \frac{1}{\sqrt{2}}(ds-sd) \\
\mathbf{6} & & \overline{\mathbf{3}} \\
3 \otimes 3 & = & 6 \oplus \overline{3}
\end{array}
$$

- We know that the triplet is a $\overline{3}$ from its I_3 and Y:

Combining SU(3) states (a 3rd quark)

- $3 \otimes 3 \otimes 3 = 3 \otimes (6 \oplus \overline{3}) = 10_s \oplus 8_{M,S} \oplus 8_{M,A} \oplus 1$
- Start with the fully symmetric part of the **6**:

$$\begin{array}{cl} uuu & \text{3 such states} \\ \frac{1}{\sqrt{3}}(ddu + udd + dud) & \text{6 such states} \\ \frac{1}{\sqrt{6}}(dsu + uds + sud + sdu + dus + usd) & \text{1 such state} \end{array}$$

  Ten states that are fully symmetric

- Now, the mixed symmetry part of the **6**:

$$\frac{1}{\sqrt{6}}\left[(ud + du)u - 2uud\right] \quad \text{8 such states}$$

  Eight states like this

- Now on to the $\overline{3}$:

$$\frac{1}{\sqrt{6}}\left[(ud - du)s + (usd - dsu) + (du - ud)s\right] \quad \text{8 such states}$$

  Eight states like this

- Final state, totally antisymmetric

Combining SU(3) states ($q\overline{q}$)

- Start with $\pi^+ = u\,\overline{d}$
- Using:

$$\begin{aligned} I_-|\overline{u}\rangle &= -|\overline{d}\rangle \\ I_-|\overline{d}\rangle &= +|\overline{u}\rangle \end{aligned}$$

We find:

$$\begin{aligned} I_-|u\overline{d}\rangle &= -|uu\rangle + |dd\rangle \\ &= \sqrt{2}\,|I=1\; I_3=0\rangle \\ \pi^0 &= \frac{1}{\sqrt{2}}(|d\overline{d}\rangle - |u\overline{u}\rangle) \end{aligned}$$

Doing this again: $\pi^- = d\,\overline{u}$

- Now add strange quarks: 4 combinations

$$\begin{array}{cccc} u\overline{s} & d\overline{s} & \overline{u}s & \overline{d}s \\ K^+ & K^0 & K^- & \overline{K^0} \end{array}$$

- One missing combination:

$$(d\overline{d} + u\overline{u} - 2s\overline{s})/\sqrt{6} \quad \equiv \eta'$$

These 8 states are called an octet

- One additional independent combination: the singlet state

$$(u\overline{u} + d\overline{d} + s\overline{s})/\sqrt{6}$$